

Phase transition in the 2d random Potts model in the large- q limit

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Phase transition in the two-dimensional q -state Potts model with random ferromagnetic couplings in the large- q limit is conjectured to be described by the isotropic version of the infinite randomness fixed point of the random transverse-field Ising spin chain. This is supported by extensive numerical studies with a combinatorial optimization algorithm giving estimates for the critical exponents in accordance with the conjectured values: $\beta = (3 - \sqrt{5})/4$, $\beta_s = 1/2$ and $\nu = 1$. The specific heat has a logarithmic singularity, but at the transition point there are very strong sample-to-sample fluctuations. Discretized randomness results in discontinuities in the internal energy.

Quenched disorder could often change the properties of phase transitions but exact information is scarce about the singularities in random fixed points, in particular about isotropic classical systems with short range interactions. For most of random classical systems - such as in spin glasses and random ferromagnets - the critical behavior is governed by conventional random fixed points (CRFP-s), in which the strength of disorder remains finite under renormalization.

There is, however, a class of strongly anisotropic two-dimensional (2d) random models, in which the disorder is strictly correlated in 1d and there are exact results about their critical behavior. These models, such as the McCoy-Wu model [1], are closely related to random quantum spin chains, and their critical behavior is usually governed by an infinite randomness fixed point [2] (IRFP), in which - under renormalization - the strength of disorder grows without limits. In an IRFP the singularities are primarily determined by disorder effects and for almost every studied, strongly disordered, 1d quantum (or 2d classical) systems the critical exponents are the same as in the universality class of the random transverse-field Ising model [3] (RTIM).

Having in mind the known relation between the critical behavior of 2d classical and corresponding 1d quantum systems [4] with non-random couplings one might ask the question if a similar correspondence exists for random systems and in particular what is the 2d isotropic classical counterpart of the RTIM. Some hints about a possible 2d isomorphism of the RTIM is presented in Ref. [5] in which the operator profiles of the RTIM are found to be conformally invariant, the property of which should be shared with some 2d isotropic random system, which - by definition - could obey conformal symmetries [6].

In this Letter we conjecture that the possible candidate for this rôle is the q -state Potts model [7] with random ferromagnetic bonds (random bond Potts model - RBPM) in the large- q limit and present the following arguments. First, we note that in the above limit the high-temperature expansion of the RBPM is dominated by a single diagram [8], \mathcal{F} , so that the critical behavior is primarily determined by disorder effects as in the IRFP

of the RTIM. Second, we have performed extensive numerical calculations, in which \mathcal{F} is exactly determined by a very efficient combinatorial optimization algorithm [9] and the obtained bulk and surface magnetization scaling dimensions, x and x_s , respectively, are found in good agreement with the corresponding exact values for the RTIM [3]:

$$x = \frac{3 - \sqrt{5}}{4}, \quad x_s = \frac{1}{2}. \quad (1)$$

Third, we point out on apparent topological similarities between the ground state of the RTIM and the fractal structure of the optimal graph, \mathcal{F} , which could then explain the isomorphism between the two problems.

In the following we introduce the RBPM by the Hamiltonian:

$$-\frac{H}{kT} = \sum_{\langle ij \rangle} K_{ij} \delta(\sigma_i, \sigma_j), \quad (2)$$

where the spin variable at site i is $\sigma_i = 0, 1, \dots, q-1$, $K_{ij} > 0$ are random ferromagnetic couplings and the summation runs over nearest neighbor pairs. In the random cluster representation the partition function of the system is expressed in terms of $v_{ij} = \exp K_{ij} - 1$ as [8]

$$Z = \sum_F q^{C(F)} \prod_{ij \in F} v_{ij}, \quad (3)$$

where the summation runs over all subset of bonds, F , and $C(F)$ is the number of connected components of F , counting also the isolated sites. Having the parameterization, $v_{ij} = q^{\alpha_{ij}}$ the partition function is expressed as

$$Z = \sum_F q^{f(F)}, \quad f(F) = C(F) + \sum_{ij \in F} \alpha_{ij}, \quad (4)$$

which in the large- q limit is indeed dominated by the largest contribution, $f^* = \max_F f(F)$, and the partition function is asymptotically given by $Z = Nq^{f^*}$ where the number of optimal sets (OS-s), N , is likely to be one.

One of the few rigorous results about random systems is due to Aizenman and Wehr [10] who states that in the

2d RBPM the internal energy is continuous at the phase transition point for any q , if the probability distribution of the couplings is absolutely continuous. In numerical calculations [11,12] one usually studies the properties of the system at the phase transition point, which is known by duality [13], and universal, i.e. disorder independent, critical behavior has been observed even for atomistic (c.f. bimodal) distributions. The scaling dimension, x , is a monotonously increasing function of q , but its saturation value in the large- q limit is difficult to be estimated due to strong logarithmic corrections [14]. The correlation length exponent is close to $\nu = 1$, for any value of q . Note, however, that by transfer matrix calculation Jacobsen and Cardy [12] have obtained $\nu < 1$ for the bimodal distribution of disorder, which violates the rigorous upper bound, $\nu \geq 2/d$ [15].

In the large- q limit the critical properties of the RBPM are related to the structure of the clusters in the OS in close analogy with percolation [16]. In the paramagnetic phase there are only finite clusters in the OS and the linear extent of the largest clusters is used to define the correlation length, ξ . In the ferromagnetic phase there is an infinite cluster and the ratio of lattice points belonging to it is related to the (finite) magnetization of the RBPM. Finally, at the phase transition point the largest (infinite) cluster is a fractal, and its fractal dimension, d_f , is related to the magnetization scaling dimension as $d = d_f + x$. Similarly, the sum of the surface fractal dimension of the percolating cluster, d_f^s , and the anomalous dimension of the surface magnetization, x_s , gives the Euclidean dimension of the surface: $d^s = 1 = d_f^s + x_s$.

In Ref. [8] the OS was approximatively calculated by the simulated annealing method. Results, obtained on relatively small lattices (up to 24×24), such as $x = 0.17 - 0.19$ were consistent with the conjectured value in Eq.(1), however, due to the relatively large error and also due to the lack of precise estimates about another exponents we could not make a definite statement about the universality class of the transition.

In the present Letter we apply a recently developed combinatorial optimization algorithm [9] with which we can determine the *exact* OS in strongly polynomial time, i.e. the time of computation does not depend on the form of the disorder. With this algorithm we could treat far larger systems as before, averages and distributions were calculated for systems with $L = 32, 64, 128$ and 256 over at least thousand disorder realizations. (We have also analyzed a few samples of 512×512). As a consequence our estimates of the critical exponents become much more accurate than before and we studied, at the first time, also the surface properties of the RBPM.

In most of the calculation the couplings and thus the parameters, α_{ij} , were taken from a bimodal distribution: $P(\alpha) = [\delta(w + \Delta w - \alpha) + \delta(w - \Delta w - \alpha)]/2$ with $w > \Delta w > 0$, so that the distance of the critical temperature, t , is measured by $t \approx 1 - 2w$. Having the parameter $\Delta w = 1/3$ the microscopic length-scale in the problem

[8], $l_c \approx (2\Delta w)^{-2}$, was not too large. We also used the continuous (uniform) distribution: $P_u(\alpha) = 1/(2u)$, for $0 \leq \alpha < u$ and zero otherwise. Here the distance from the critical point is given by $t \approx 1 - u$. To calculate bulk (surface) quantities we applied periodic (open) boundary conditions (b.c.-s).

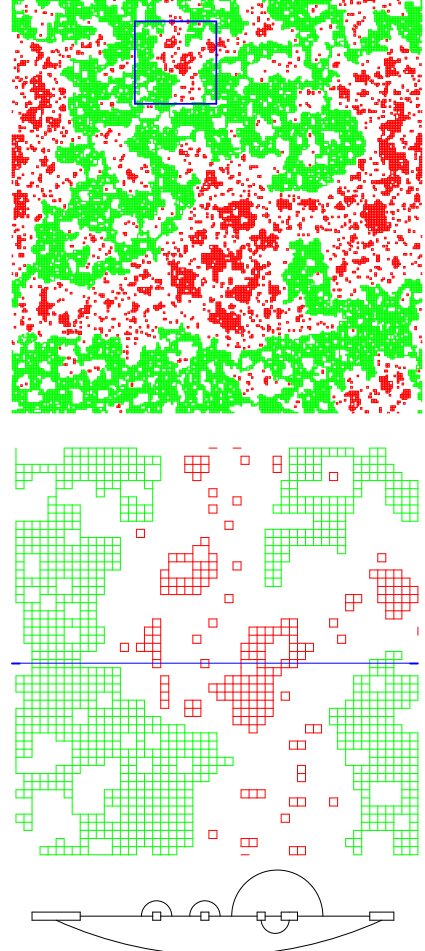


FIG. 1. top: OS for a typical disorder realization at the critical point on a 256×256 lattice with periodic boundary conditions. Percolating and finite clusters are marked with gray (green) and dark (red) lines, respectively. middle: Enlargement of a square proportion of size 48×48 in the upper-middle part of the OS to illustrate self-similarity. bottom: The connectivity structure of the OS along a line, which consists of six connected units (“spins”) and five open units (“bonds”).

First, we considered the behavior of the system at the transition point, when the cluster structure of a typical OS is shown in Fig. 1. As illustrated in the middle of Fig. 1 the OS is self-similar and its topology, being isotropic, can be conveniently represented by the connectivity structure (CS) of the OS at a given line, as shown in the bottom of Fig. 1. We are going to analyze the CS after presenting the numerical results.

We have checked that asymptotically half of the sites

of the OS are isolated: the probability $P(L)$ that a given site belongs to a cluster having two or more sites, behaves like $P(L) \simeq 1/2 (1 + 1/(2 \ln L))$. We have also checked that the largest connected cluster is indeed a fractal (and not a multi-fractal). Its fractal dimension, d_f , is calculated by direct dimensional analysis and by finite-size scaling of the average mass of the largest cluster. Here we present an analysis of the probability distribution function, $R(m, L)$, which measures the fraction of clusters having a size at least m and which, according to scaling theory [16], should asymptotically behave as:

$$R(m, L) = m^{-\tau} \tilde{R}(m/L^{d_f}), \quad (5)$$

with $\tau = (2 - d_f)/d_f$. With the conjectured value of $d_f = 2 - x = (5 + \sqrt{5})/4 = 1.809$ we obtained a very good scaling collapse, which is shown in Fig. 2.

To characterize the accuracy of the collapse we have measured the surface of the overlap of the scaled curves with varying value of d_f . As shown in the inset a) to Fig. 2 the best collapse is indeed around the conjectured value, so that our estimate $x = 0.190(5)$ has a relatively small error.

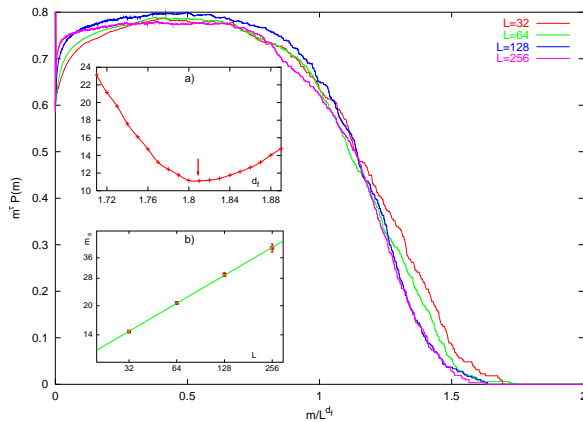


FIG. 2. Scaling collapse of the reduced probability distribution function, $R(m, L)m^\tau$, in Eq.(5) with $d_f = (5 + \sqrt{5})/4 = 1.809$. Inset a): Surface of the collapse region for different values of d_f . The arrow shows the conjectured value. b) Average mass of surface sites of the largest cluster vs. linear size of the system in a log-log plot. The slope of the straight line corresponds to $d_f^s = 0.495(10)$.

Next, we considered the surface magnetization properties of the model by using open b.c.-s and measuring the average mass of surface sites of the largest cluster, m_s . As shown in the inset b) to Fig. 2 it has an asymptotic size dependence $m_s \sim L^{d_f^s}$, with $d_f^s = 0.495(10)$, which is in very good agreement with the conjectured value of $d_f^s = 1 - x_s = 1/2$, see Eq.(1).

In the following we analyze in more details the fractal structure of the OS and point out the topological similarities with the ground state wave function of the RTIM, which could explain the appearance of the same critical exponents in the two problems. As already noted in the

bottom of Fig.1 the CS consists of connected units (CU-s) (corresponding to spins in the RTIM) of variable length, l_s , and moments, μ , and of open units (OU-s) (corresponding to bonds in the RTIM) of variable length, l_b . If two neighboring CU-s, with parameters l_s^1, μ^1 and l_s^2, μ^2 , separated with an OU of l_b , belong to the same cluster, it is merged to an effective CU (represented by a connecting line in Fig. 1), with length $l_s^{12} = l_s^1 + l_s^2 + l_b$ and moment $\mu^{12} = \mu^1 + \mu^2$. This is precisely equivalent of a strong bond decimation in the RTIM, which is one ingredient of the strong disorder renormalization group (SDRG) method [17,3]. Similarly, if a CU with l_s and μ and with neighboring bonds of lengths: l_b^1 and l_b^2 , is isolated, it does not contribute to any larger cluster, therefore - at larger length-scales - can be eliminated (represented by an overgoing line in Fig. 1) and the new effective bond has a length of $l_b^{12} = l_b^1 + l_b^2 + l_s$. This process then equivalent of a strong field decimation in the SDRG for the RTIM. Thus we can conclude that for any CS of the OS in the RBPM one can construct an equivalent ground state of the RTIM, and one can give a set of couplings, J_i , and transverse fields, h_i , for which the given ground state is realized. We recall that in the RTIM the bulk (surface) magnetization is related to the average moment of a bulk (surface) effective spin [3] in close analogy with the computation of the same quantities in the OS of the RBPM. If we now assume that in the two problems the statistics of the appearance of states with equivalent topology is also similar, we arrive to the conjectured relation about the critical exponents in Eq.(1).

In the following we turn to study the energy-density of the model outside the transition point. To do this one should determine the OS at different temperatures, however, in a finite system of size, L , there are only a finite number of different OS-s, their typical number being L , independently of the type of disorder. This result can be interpreted that two neighboring OS-s differ in average by one line ($\sim L$) of edges. For a given sample the free energy is a piece-wise linear function of t and the internal energy is a step-like function having typically L steps. Averaging over disorder the average internal energy becomes continuous for continuous distributions (see. Fig.3), whereas for discrete distributions some discontinuities, located at special isolated points, remain. (The discontinuous behavior of the internal energy in this case is somewhat analogous to that of the magnetization of the random-field Ising model at $T = 0$ having the same bimodal distribution [18].) There is a discontinuity at the phase-transition (self-dual) point, as we illustrate with the sequence of finite-size latent heats: $[\Delta E(32)]_{av} = 0.0456(90)$, $[\Delta E(64)]_{av} = 0.0477(52)$, $[\Delta E(128)]_{av} = 0.0484(28)$ and $[\Delta E(256)]_{av} = 0.0474(14)$ for the bimodal distribution with $\Delta w = 1/3$, which approach a finite value in the thermodynamic limit. The discontinuities in the internal energy are due to degeneracies, which are connected also to finite clusters. For example at the self-dual point any lattice point which

has two couplings - one weak and one strong - to an existing cluster, could either be connected or disconnected. Thus the jumps in the internal energy are related not exclusively to the largest cluster. The contribution of the largest cluster only, which has a diverging size ξ , is expected to cause singularity in the specific heat at the two sides of the transition point.

This true singularity of the specific heat, which is expected to be independent of the type of disorder, will be investigated numerically in the following. Here, as usual, the thermodynamic limit should be taken first and then approach the transition point. In our numerical study we observed that close to the transition point, in the order of $|t| \sim 1/L$, there are large sample to sample fluctuations. As an illustration in the inset a) to Fig.3 we have shown the distribution of the finite-size specific heat, C_v^1 , defined for a given sample as the ratio of the distance between the first two energy steps and the corresponding temperature difference, just at the right of the transition point. The distribution, as shown in the inset has a broad, power-law tail and its second moment does not exist. This fact represents the strong randomness character of the transition. Using continuous distribution of disorder the strong fluctuation regime close to the transition point remains qualitatively the same. Therefore, to determine the thermodynamic singularity of the specific heat we analyzed its behavior for $|t| > 1/L$. As presented in the inset b) to Fig.3 the specific heat has a logarithmic singularity of the form of $C_v(t) \sim (\ln|t|)^\epsilon$, with $\epsilon \leq 1$. Thus the specific heat exponent is $\alpha = 0$ and the correlation length exponent is $\nu = 1$. This latter result is the half of the similar exponent of the RTIM in the strongly anisotropic IRFP [3].

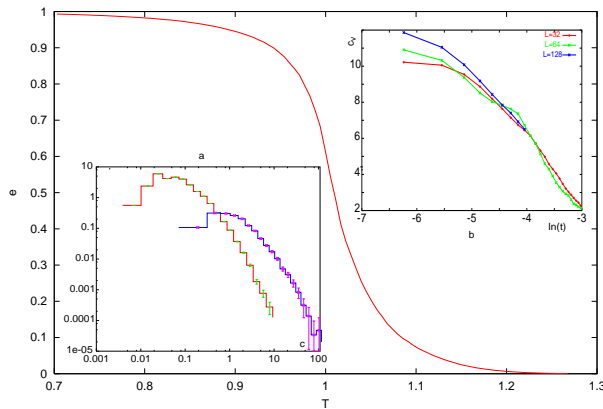


FIG. 3. Average internal energy for the continuous distribution. Inset a): distribution of the finite-size latent heat at the right side of the transition point for the bimodal distribution for $L = 32$ (left) and $L = 128$ (right). Inset b): the average specific heat for the continuous distribution as a function of $\ln|t|$ in the region $Lt \gg 1$.

To conclude our investigations have shown that the

critical behavior of the RBPM in the large- q limit is dominated by strong disorder effects and possibly related to the IRFP of the RTIM. The conjectured exact values of the critical exponents are checked by extensive numerical calculations. If an asymptotically exact RG treatment - in the same spirit as the SDRG for the RTIM - can be constructed, should be clarified by future research.

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